AD-A091 206 DIRECTORATE OF AEROSPACE STUDIES KIRTLAND AFB NM F/G 19/4
THE PROBABILITY OF HIT OF THE ANTIARMOR CLUSTER MUNITION.(U)
JUN 79 C L BOHN
UNCLASSIFIED DAS-WP-79-4 NL

1 OF 1
AD A
091206

END
DATE
FILMED
12-80
DTIC

AS-WP-79-4

DAS-WP-79-4

LEVEL II

12

AD A091206

THE PROBABILITY OF HIT OF THE ANTIARMOR CLUSTER MUNITION

JUNE 1979

2D LT COURTLANDT L. BOHN, USAF

Changes p. 3

DIRECTORATE OF AEROSPACE STUDIES
DCS/PLANS AND PROGRAMS, HQ AFSC
KIRTLAND AFB, NEW MEXICO 87117

80 10 21 091

ABSTRACT

One weapon system concept proposed as a Wide-Area Antiarmor Munition (WAAM) is the Antiarmor Cluster Munition (ACM). The ACM consists of a submunition dropped to the battlefield by parachute; during its descent the system behaves like a damped spherical pendulum. At a certain preprogrammed height (nominally 1m) the submunition detonates, sending forth discrete warheads at hypervelocities. This method of delivery gives rise to certain ramifications regarding the probability that a warhead will hit a target. Specifically, the pendular motion acts to degrade the probability of hit P_H. This paper presents a formal derivation of P_H with regard to hitting a solitary target; the treatment entails coupling the physics of the pendulum to geometrical considerations. It also presents numerical examples which show P_H is quite sensitive to the (ensemble-averaged) maximum amplitude of the pendular motion.

I give special thanks to Bruno J. Manz for bringing this problem to my attention and for several helpful discussions. I also thank Eileen P. Yankee for preparing the manuscript.

COURTLANDT L. BOHN, 2d Lt, USAF
Staff Physicist
Directorate of Aerospace Studies

This paper has been reviewed and is approved for publication.

HARRY L. GOGAN
Technical Director
Directorate of Aerospace Studies

RUPERT A. LEONARD, JR, Lt Col, USAF
Director of Aerospace Studies
DCS/Plans and Programs, HQ AFSC

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: (14) DAS-WP-79-4	2. GOVT ACCESSION NO.: AD-A091 306
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): (6) THE PROBABILITY OF HIT OF THE ANTIARMOR CLUSTER MUNITION	5. TYPE OF REPORT & PERIOD COVERED: (9) FINAL
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): (12) COURTLANDT L. BOHN	8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Directorate of Aerospace Studies, Kirtland AFB NM 87117	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE: (11) June 1979
	13. NUMBER OF PAGES: 42 (12) 40
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE
DISTRIBUTION UNLIMITED

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

ANTIARMOR CLUSTER MUNITION, PROBABILITY OF HIT,

WIDE-AREA ANTIARMOR MUNITIONS

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

One weapon system concept proposed as a Wide-Area Antiarmor Munition (WAAM) is the Antiarmor Cluster Munition (ACM). The ACM consists of a clustered bomb unit (CBU) dropped to the battlefield by parachute; during its descent the system behaves like a damped spherical pendulum. At a certain preprogrammed height (nominally 1m) the CBU detonates, sending forth discrete warheads at hypervelocities. This method of delivery gives rise to certain ramifications regarding the probability that a warhead will hit a target. Specifically, the pendular motion acts to degrade the probability of hit P_H. This paper presents

DD FORM 1473, 1 JAN 73 — EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

404227

20. ABSTRACT (Continued)

a formal derivation of P^{Π}_{H} with regard to hitting a solitary target; the treatment entails coupling the physics of the pendulum to geometrical considerations. It also presents numerical examples which show P^{Π}_{H} is quite sensitive to the (ensemble-averaged) maximum amplitude of the pendular motion. -

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

I. INTRODUCTION

One of the areas receiving widespread attention in the Air Force today is the development of Wide-Area Antiarmor Munitions (WAAMs). These are typically clustered or dispenser carried munitions with large ground-plane kill areas and are to be used in conventional warfare. Several weapon system concepts have been proposed for WAAM applications, one of which is the so-called Antiarmor Cluster Munition (ACM). This paper presents a formal calculation of the probability of hit of the ACM.

The ACM concept involves the delivery of clustered bomb units (CBUs) to the battlefield via parachute. Each CBU is to consist of a discrete number of antiarmor warheads, or "slugs", to be explosively dispensed at hypervelocity once the CBU reaches a preprogrammed height (nominally 1m). The exact configuration of the CBU has not yet been firmly established; for concreteness, we assume it is shaped like an equilateral triangle and carries three slugs. Figure 1 portrays the conceptual ACM configuration under study.

Once the parachute is deployed, the system behaves like an oscillator -- a pendulum. Thus, at the point of firing, the system generally will not be oriented vertically. For this reason the slugs may fly over or impact the ground short of targets, and therefore the pendular motion acts to degrade the probability of hit P_H. As we will point out, the behavior of P_H is quite sensitive to the orientation of the CBU.

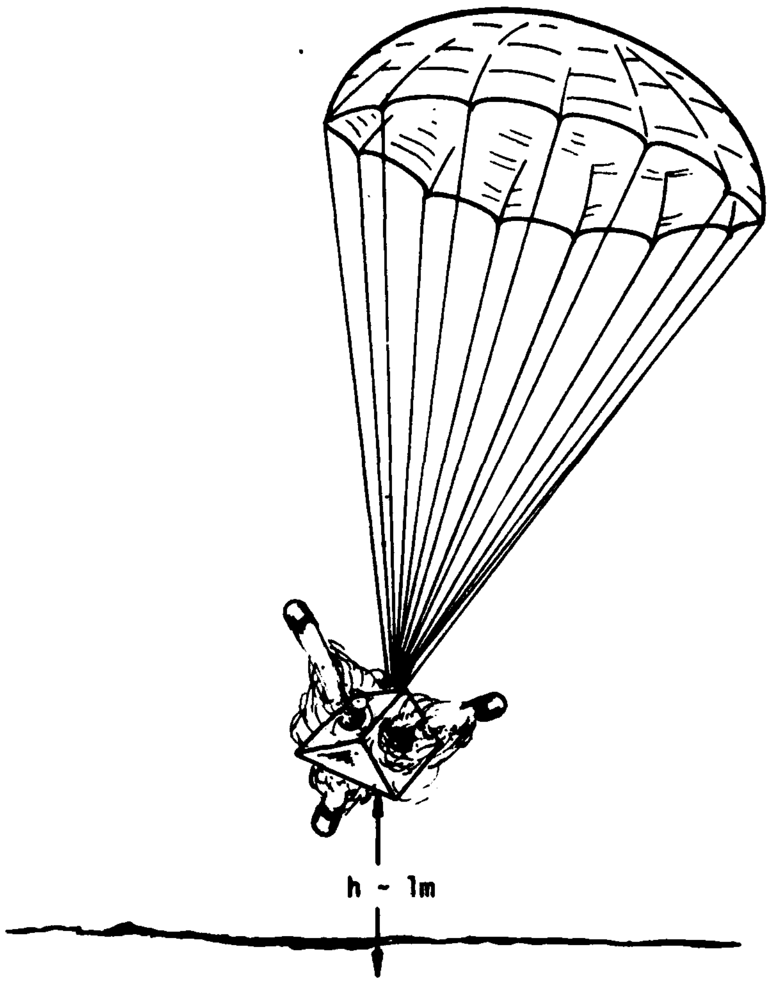

Figure 1. Conceptual Make-up of the ACM

Our goal is to formally derive P_H. To make the calculations tractable, we set forth the following assumptions:

1. The system always damps to simple harmonic motion (SHM) before firing takes place; however, the damping constant itself is small. This means damping over times as small as one period of the oscillation is negligible, but cumulative damping over times as large as the time of descent is substantial.

2. The (three-faced) submunition always fires from height h attempting to hit a single target randomly located on a circle of radius R centered on the submunition.

3. The slugs travel in approximately a straight line over their effective range.

4. The target is situated on flat, open terrain.

5. The target is a sphere of radius h, and therefore its projection with respect to the submunition is always the same.

Assumptions 2, 4, and 5 describe a somewhat contrived situation; however, they do not impact on the general properties of P_H.

As presented, the problem is one of applied classical mechanics, and its solution evolves from the physics of the pendulum coupled to geometric considerations. The physics of the pendulum governs the equations of motion of the weapon orientation Θ, and these, in turn, tell how the orientations are distributed over a complete oscillation; that is, they determine the distribution function $f(\Theta)d\Theta$. The geometry determines the probability of hit P_H (R, h, Θ, ϕ) given the weapon orientation Θ and the target location ϕ which obeys a random distribution function $g(\phi)d\phi$. Once P_H (R, h, Θ, ϕ) is known, the calculation becomes purely mathematical; the solution arises via an integration over the two distribution functions and becomes a function of the maximum amplitude Θ_m.[1] Finally, an integration over a cluster of ACMs (which serves as an ensemble), each member of which is deployed using only one technique (for example, deployed from the same altitude by the same type of munitions dispenser), completes the calculation. In sections II and III these steps are developed formally. Section II treats the physics of the pendulum, and section III presents the major aspects of the calculation. Examples of the numerical results are given in section IV, as are the general conclusions. Detailed derivations are reserved for the appendices.

---

1. P_H also is generally a function of the minimum amplitude Θ_o, but we will explicitly treat only cases for which $\Theta_o=0$ or $\Theta_o=\Theta_m$.

II. PHYSICS OF THE PENDULUM

To calculate P_H (R, h, Θ, ϕ) we need only be concerned with the state (orientation) of the ACM at the time of firing. In averaging the orientation Θ over its distribution function $f(\Theta)d\Theta$, an integration over a single period is required. At the time of firing, the ACM is assumed to be executing SHM with negligible damping over a single period. Hence, consistent with our assumptions, we need only to treat the ACM formally as an undamped pendulum. However, the oscillations will generally not be confined to a plane; rather, they will generally be "spherical". Thus, the subject of this section is the undamped spherical pendulum executing SHM, and the goal is to derive the corresponding distribution function $f(\Theta)d\Theta$.

We begin by deriving the equations of motion using the well-known Lagrangian formalism of classical mechanics.[1] The spherical pendulum is shown in figure 2; the submunition of mass m is always confined to a sphere of radius ℓ (the length of the parachute) and is subject to the gravitational acceleration $\vec{g}$. At time t, the system is in the state defined by the amplitude $\Theta(t)$ and the azimuth $\gamma(t)$. The kinetic energy T and potential energy V then follow immediately by inspection of the figure:

$$T = \frac{1}{2} mv^2 = \frac{1}{2} m\ell^2 \left[\left(\frac{d\Theta}{dt} \right)^2 + \sin^2\Theta \left(\frac{d\gamma}{dt} \right)^2 \right] \tag{1a}$$

$$V = - mg\ell \cos\Theta \tag{1b}$$

The Lagrangian L is just L=T-V, or

$$L = \frac{1}{2} m\ell^2 \left[\left(\frac{d\Theta}{dt} \right)^2 + \sin^2\Theta \left(\frac{d\gamma}{dt} \right)^2 \right] + mg\ell \cos\Theta \tag{2}$$

1. The equations of motion are derived in several textbooks, such as Slater, J.C., and Frank, N.H., Mechanics, pp. 79-86, McGraw-Hill Book Company, Inc., New York, 1947.

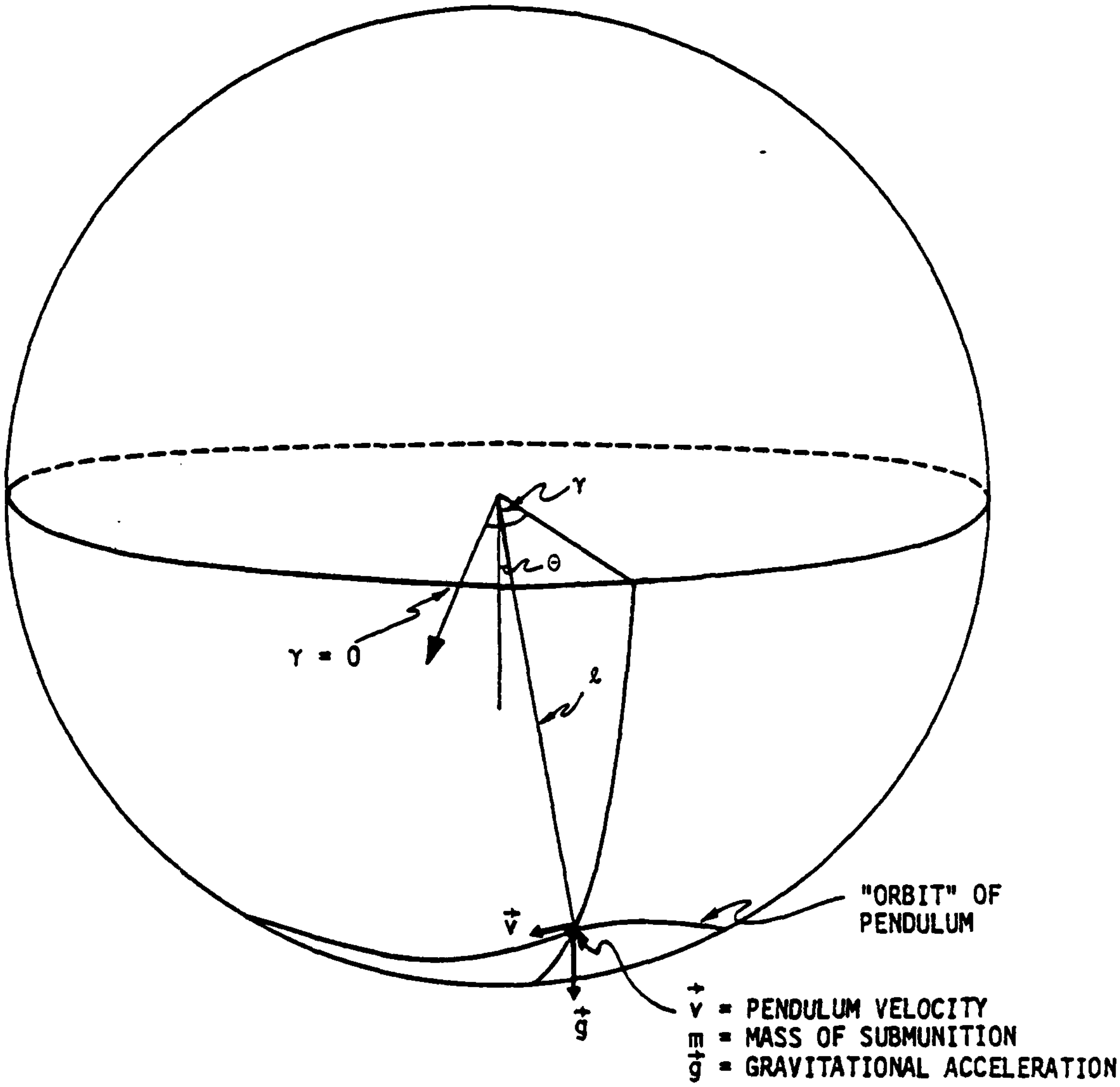

Figure 2. The Spherical Pendulum

From the Lagrangian follow the equations of motion; they correspond to

$$\frac{d}{dt}\left[\frac{\partial L}{\partial(d\Theta/dt)}\right] - \frac{\partial L}{\partial\Theta} = 0 \tag{3a}$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial(d\gamma/dt)}\right] - \frac{\partial L}{\partial\gamma} = 0 \tag{3b}$$

By substitution, we find

$$\frac{d^2\Theta}{dt^2} - \sin\Theta\cos\Theta\left(\frac{d\gamma}{dt}\right)^2 + \frac{g}{\ell}\sin\Theta = 0 \tag{4a}$$

$$\sin^2\Theta\,\frac{d\gamma}{dt} = C \tag{4b}$$

where C is an integral of the motion. The first equation (4) simply says energy is conserved, and the second says angular momentum is conserved.

Equations (4) are "exact", but now we incorporate the SHM assumption which says $\Theta<<1$ so that $\sin\Theta \approx \Theta$ and $\cos\Theta \approx 1$. Then the equations reduce to

$$\frac{d^2\Theta}{dt^2} - \left[\left(\frac{d\gamma}{dt}\right)^2 - \frac{g}{\ell}\right]\Theta = 0 \tag{5a}$$

$$\Theta^2\,\frac{d\gamma}{dt} = C \tag{5b}$$

and upon inserting equation (5b) into equation (5a), a single equation of motion for Θ follows:

$$\frac{d^2\Theta}{dt^2} + \frac{g}{\ell}\Theta - \frac{C^2}{\Theta^3} = 0 \tag{6}$$

The solution of this nonlinear differential equation cannot be written in terms of elementary functions; however, it becomes greatly simplified for two special cases - (1) γ = constant, and (2) Θ = constant. For case (1),

equations (5) reduce to

$$\frac{d^2\Theta}{dt^2} + \frac{g}{\ell}\Theta = 0 \rightarrow \Theta(t) = \Theta_m \sin \omega t \tag{7}$$

where Θ_m is the maximum amplitude and $\omega = \sqrt{g/\ell}$ is the circular frequency. This is just the equation of motion of a simple, i.e., planar, pendulum; the orbit in figure 2 is then an arc. For case (2), equations (5) reduce to

$$\left(\frac{d\gamma}{dt}\right)^2 - \frac{g}{\ell} = 0 \rightarrow \gamma(t) = \omega t \tag{8}$$

This is the equation of motion of a "conical" pendulum which executes a circular orbit on the sphere in figure 2.

Because the probability of hit P_H must increase as Θ decreases, cases (1) and (2) in fact bound the calculations in that P_H is maximized for case (1) and minimized for case (2). Thus, we will focus our attention on these two cases alone. The derivation of the distribution function $\tilde{f}_C(\Theta)d\Theta$ for the perturbed conical pendulum is offered in appendix 1, and it can be folded into the discussion using the same techniques presented in section III.

We now turn to developing the distribution functions for the simple and conical pendulums. Equation (7) tells us how to form the distribution function $f_S(\Theta)d\Theta$ for the simple pendulum. The variable which determines the state of the system is the time t, and the distribution function for t normalized over a full period $\tau = 2\pi/\omega$ is simply

$$f(t)dt = \frac{dt}{\tau} \tag{9}$$

But according to equation (7),

$$t = \frac{1}{\omega} \arcsin\left(\frac{\Theta}{\Theta_m}\right) \tag{10}$$

so that

$$dt = \frac{\tau}{2\pi} \frac{d\Theta}{\sqrt{\Theta_m^2 - \Theta^2}} \tag{11}$$

Hence, equation (9) becomes

$$f_S(\Theta)d\Theta = \frac{1}{2\pi} \frac{d\Theta}{\sqrt{\Theta_m^2 - \Theta^2}} \tag{12}$$

For the conical pendulum, the distribution function $f_C(\Theta)d\Theta$ is simply a delta function because $\Theta = \Theta_m$ is a fixed constant:

$$f_C(\Theta)d\Theta = \frac{1}{2\pi} \delta(\Theta - \Theta_m)d\Theta \tag{13}$$

Both $f_S(\Theta)$ and $f_C(\Theta)$ are normalized over a full period:

$$4\int_0^{\Theta_m} f_S(\Theta)d\Theta = 4\int_0^{\Theta_m} f_C(\Theta)d\Theta = 1 \tag{14}$$

III. THE PROBABILITY OF HIT

The probability of hit P_H (R, h, Θ, ϕ) given the orientation Θ of the ACM and the azimuth ϕ of the spherical target arises solely from geometrical considerations. The calculation of the expected probability of hit P_H $(R, h, <\Theta_m>)$ involves integrations over the random distribution function $g(\phi)d\phi$ and the (not random!) distribution function $f(\Theta)d\Theta$ derived in the last section. Thus, we begin this section with a discussion of the geometry and then proceed with the calculations.

A. GEOMETRY OF THE ACM ATTACK

When the submunition fires, its plane of fire generally is not parallel to the plane of the ground. This means that in certain cases, i.e., for certain R and Θ, the slug may fly over the target or may impact the ground short of the target. This situation is portrayed in figure 3 which shows the weapon's plane of fire is inclined by the angle Θ to the "equator" which runs parallel to the ground and through the submunition firing at height h. The figure makes clear that the target dimensions give rise to a "lethal" region bounded by the "critical azimuths" $\pm\phi_{CR}$. (There are actually two lethal regions which are diametrically opposed.) If $\Theta \leq \alpha/2$, the angular extent of the target, then the target certainly lies within the plane of fire, i.e., $\phi_{CR} = \pi/2$. The probability that a slug hits a target outside the lethal region is zero.

As figure 3 implies, the calculations involve spherical geometry because, from the submunition's viewpoint, it is the target's projection on the sphere of radius R (corresponding to the range) that is being shot at. The angle subtended by a spherical target of radius h at range R is illustrated in figure 4 from which we infer

$$\alpha = 2 \arcsin\left(\frac{h}{R}\right) \tag{15}$$

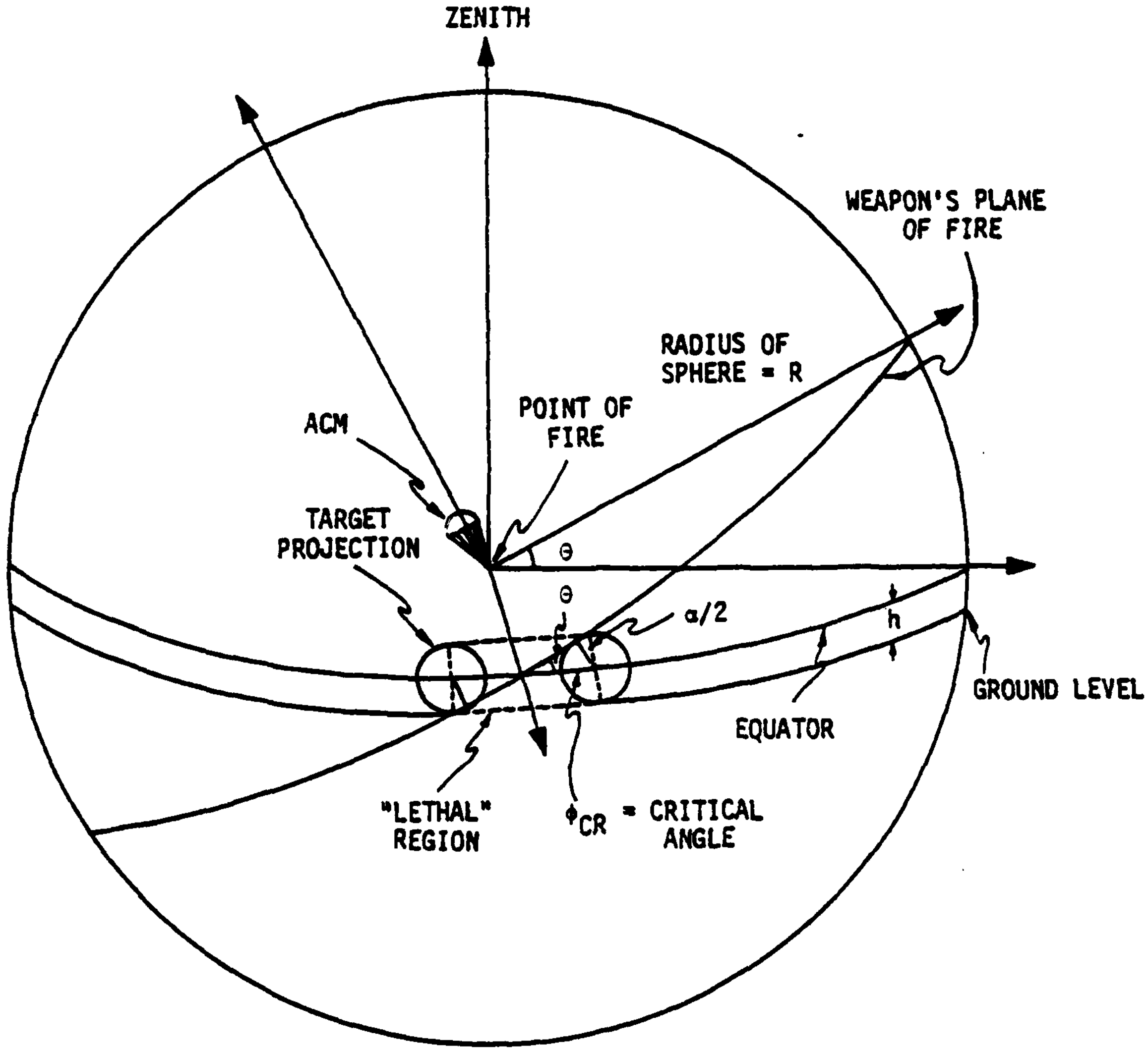

Figure 3. Geometry of the ACM Attack

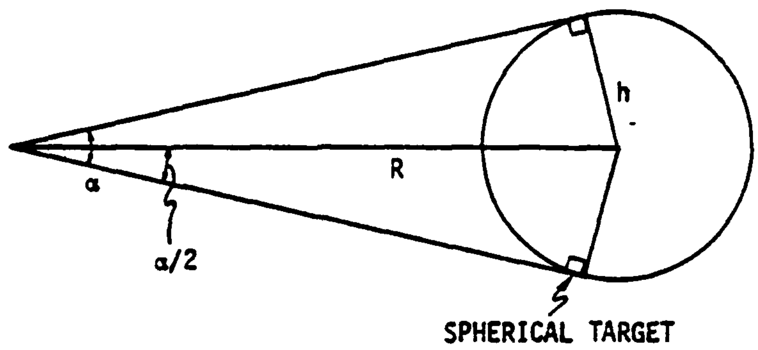

Figure 4. Target Geometry

Obviously, for $\Theta=0$, the probability of hitting a single target at range R, assuming no other targets or obstacles lie inside the circle of radius R, is simply

$$P_H\ (R,\ h,\ \Theta = 0) = \frac{3\alpha}{2\pi} = \frac{3}{\pi}\ \arcsin\left(\frac{h}{R}\right) \tag{16}$$

Here, the factor 3 arises because we have assumed the submunition simultaneously fires three slugs at azimuths separated by 120°. Clearly, P_H drops from its maximum value, unity, at $R=h/\sin(\pi/3)$.

The geometry of the lethal region as viewed from the submunition is lifted out of figure 3 and shown in figure 5. If the target lies inside $\pm\phi_{CR}$, then there is a non-zero chance for a hit, but if the target lies outside this boundary, there is no chance for a hit. This is precisely what figure 5 shows.

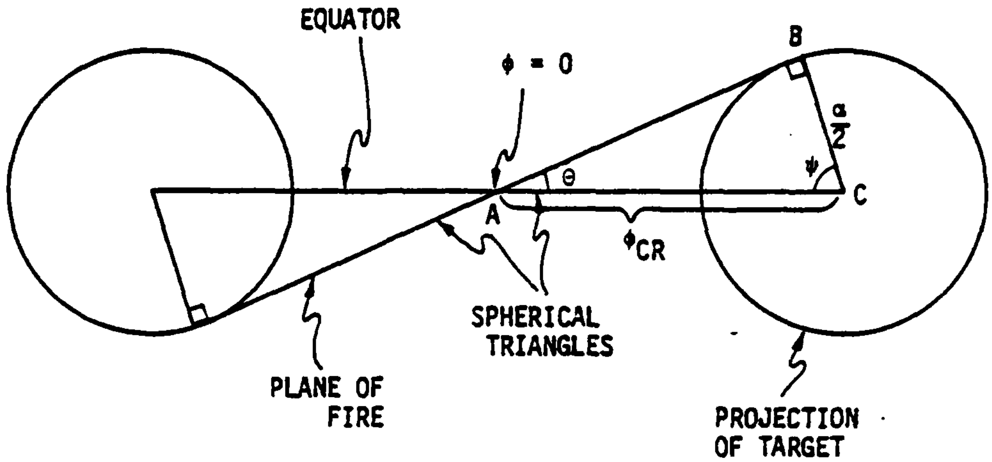

Figure 5. Geometry of the Lethal Region

We now derive ϕ_{CR} from figure 5 using the spherical triangle ABC which arises from the target dimensions and the orientation of the ACM. The angle ψ is denoted merely as an auxiliary angle to assist in the derivation. The spherical triangle relations which follow from the figure are:

$$\sin\psi = \cos\Theta \sec(\alpha/2) \tag{17a}$$

$$\cos\phi_{CR} = \cot\Theta \cot\psi \tag{17b}$$

Combining these gives

$$\phi_{CR} = \arcsin\left[\frac{\sin(\alpha/2)}{\sin\Theta}\right] \tag{18}$$

Equation (18) holds for $\Theta > \alpha/2$; if $\Theta \leq \alpha/2$, then $\phi_{CR} = \pi/2$ always. Under the SHM assumption, the angle $\alpha/2$ is always small when $\Theta > \alpha/2$, but it is not necessarily small when $\Theta \leq \alpha/2$. Thus, under the SHM assumption and according to equations (15) and (18), the critical angle is given by

$$\phi_{CR} = \begin{cases} \pi/2 & \text{for } \Theta \leq \alpha/2 = h/R \\ \arcsin(h/R\Theta) & \text{for } \Theta > \alpha/2 = \arcsin(h/R) \end{cases} \tag{19}$$

B. DERIVATION OF THE PROBABILITY OF HIT

We know that if the target lies outside the lethal region of figure 3, then the corresponding probability of hit is zero. Otherwise, however, P_H is non-zero, and we now seek to quantify this.

We begin by examining figure 6 which shows a projection of the target lying inside the lethal region. The angle exposed to the ACM's plane of fire is β, and the angles χ and ψ are auxiliary angles to assist in deriving β. Once β is known, then the probability of hit is known; it is just $3\beta/2\pi$.

We now derive β using the spherical triangles ACD and BCD in figure 6. The relations corresponding to triangle ACD are

$$\cos \chi = \cos \Theta \csc \psi \tag{20a}$$

$$\tan \psi = \cot\Theta \sec \phi \tag{20b}$$

and the relation corresponding to triangle BCD is

$$\cos (\beta/2) = \cos (\alpha/2) \sec \chi \tag{20c}$$

Combining these gives

$$\beta = 2 \arccos\left[\frac{\cos (\alpha/2)}{\sqrt{1 - \sin^2\Theta \ \sin^2\phi}}\right] \tag{21}$$

Thus, the probability of hitting a single target at range R given that the target lies at azimuth ϕ inside the lethal region, and given that the orientation of the ACM is Θ, is

$$P_H (R, h, \Theta, \phi) = \frac{3\beta}{2\pi} = \frac{3}{\pi} \arccos\left[\sqrt{\frac{1-(h/R)^2}{1-\sin^2\Theta \ \sin^2\phi}}\right] \tag{22}$$

The comments made in conjunction with equation (16) also apply here. Note that for $\Theta = 0$, equation (22) reduces to equation (16).

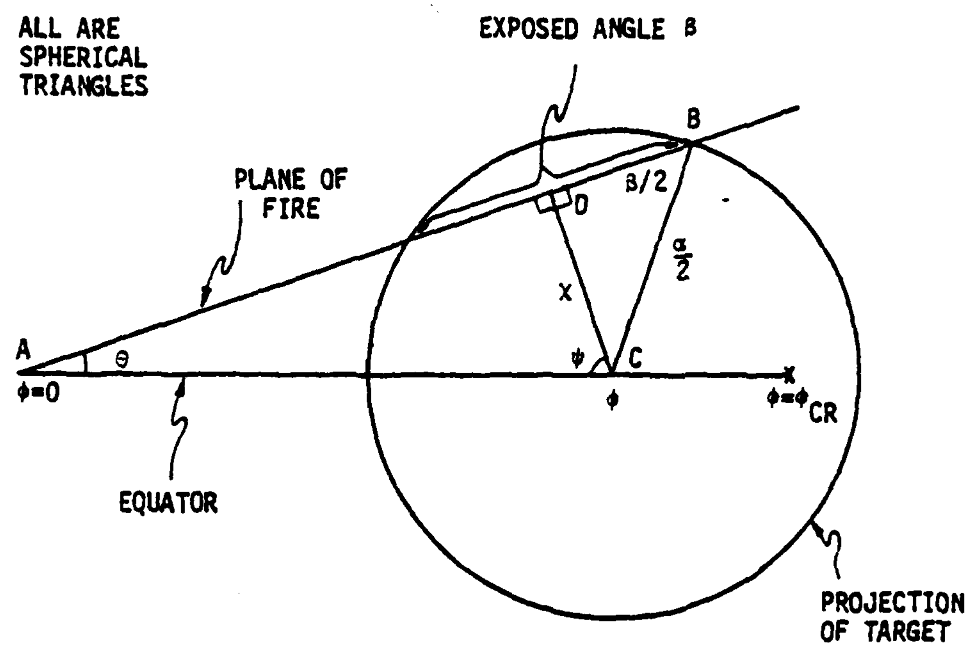

Figure 6. The Exposed Target

Next we need to average P_H (R, h, Θ, φ) over the target azimuth φ and the ACM orientation Θ; we begin by averaging over φ. Because the orientation of the submunition is random, φ is a random variable, and its distribution function $g(\phi)d\phi$ is therefore random:

$$g(\phi)d\phi = \frac{d\phi}{2\pi} \tag{23}$$

To compute P_H (R, h,Θ) requires integrating equation (22) over $g(\phi)d\phi$:

$$\begin{aligned} P_H(R, h, \Theta) &= \int_0^{2\pi} P_H(R, h, \Theta, \phi)\, g(\phi)d\phi \\ &= 4 \int_0^{\phi_{CR}} P_H(R, h, \Theta, \phi)\, g(\phi)d\phi \end{aligned} \tag{24}$$

By substitution, we have

$$P_H\ (R,\ h,\ \Theta) = \frac{6}{\pi^2} \int_0^{\arcsin\ (h/R\Theta)} \arccos \left[\sqrt{\frac{1-(h/R)^2}{1 - \sin^2\Theta\ \sin^2\phi}} \right] d\phi \qquad (25)$$

Using the SHM assumption, we can solve this integral in closed form, and we do so in appendix B. The solution corresponding to $\phi_{CR} = \pi/2$ is

$$P_H\ (R,\ h,\ \Theta \leq\ h/R) = \frac{3}{\pi} \left\{ \arccos \left[\frac{\sqrt{1-(h/R)^2}}{\cos\Theta} \right] + \right. \qquad (26a)$$

$$\left. + \frac{2h}{\pi R} \sqrt{1-\left(\frac{h}{R}\right)^2} \left[E(R\Theta/h) - \frac{\pi}{2} \sqrt{1-\left(\frac{R\Theta}{h}\right)^2} \right] \right\}$$

and the solution corresponding to $\phi_{CR}<\pi/2$ is

$$P_H\ (R,\ h,\ \theta > h/R) = \frac{6h}{\pi^2 R} \sqrt{1-\left(\frac{h}{R}\right)^2} \left[\frac{R\Theta}{h}\ E(h/R\Theta) \ - \right. \qquad (26b)$$

$$\left. -\left(\frac{R\Theta}{h} - \frac{h}{R\Theta}\right) K(h/R\Theta) \right]$$

In these equations K and E are the complete elliptic integrals of the first kind and second kind, respectively, and are elementary functions.[1]

The goal now is to average P_H (R, h, Θ) over the distribution functions $f_C(\Theta)d\Theta$ and $f_S(\Theta)d\Theta$ for the conical and simple pendulums. Because $f_C(\Theta)$ is a delta function (see equation (13)), the integration is immediate, and for the conical pendulum Θ is replaced by Θ_m in equations (26). What is left, then, is to evaluate the following integral with regard to the simple pendulum:

$$P_H^S\ (R,\ h,\ \Theta_m) = 4 \int_0^{\Theta_m} P_H\ (R,\ h,\ \Theta)\ f_S(\Theta)d\Theta \qquad (27)$$

---

1. Abramowitz, M., and Stegun, I.A., Handbook of Mathematical Functions, pp. 589-626, National Bureau of Standards, 1964. (Short Title - Abramowitz and Stegun)

The integral cannot be solved in closed form; as is shown in appendix C, the solution leaves three integrals to be evaluated numerically. For $\phi_{CR} = \pi/2$, the solution is

$$P_H^S\,(R,\ h,\ \Theta_m \le h/R) = \frac{3}{\pi}\left[\ \arccos\left(\frac{\sqrt{1-(h/R)^2}}{\cos\,\Theta_m}\right) - \right.$$

$$\left. - \frac{h}{R}\ \sqrt{1-\left(\frac{h}{R}\right)^2}\sqrt{1-\left(\frac{R\Theta_m}{h}\right)^2}\ + \frac{4h}{\pi^2 R}\ \sqrt{1-\left(\frac{h}{R}\right)^2}\ \ I_1\right] \tag{28a}$$

and for $\phi_{CR} < \pi/2$, the solution is

$$P_H^S\,(R,\ h,\ \Theta_m > h/R) = \frac{12h}{\pi^3 R}\sqrt{1-\left(\frac{h}{R}\right)^2}\ \ (I_2 + I_3) \tag{28b}$$

where

$$I_1 = \int_0^{\pi/2} E\left(\frac{R\Theta_m}{h}\ \sin\ x\right)dx \tag{28c}$$

$$I_2 = \int_0^{\arcsin\,(h/R\Theta_m)} E\left(\frac{R\Theta_m}{h}\ \sin\ x\right)dx \tag{28d}$$

$$I_3 = \int_{\arcsin\,(h/R\Theta_m)}^{\pi/2}\left\{\frac{R\Theta_m}{h}\ \sin\ x\ \left[E\left(\frac{h}{R\Theta_m \sin\ x}\right) - K\left(\frac{h}{R\Theta_m \sin\ x}\right)\right] + \right.$$

$$\left. + \frac{h}{R\Theta_m\ \sin\ x}\ K\left(\frac{h}{R\Theta_m\ \sin\ x}\right)\right\} dx \tag{28e}$$

Finally, for completeness, we must average P_H (R, h, Θ_m) over an *ensemble of ACMs, each member of which is deployed using the same technique,* to calculate P_H (R, h, $<\Theta_m>$), where $<\Theta_m>$ is the average maximum amplitude to be determined observationally. For a less restricted treatment in which the SHM assumption is not made, an integration over a Gaussian distribution of tan Θ_m would now be appropriate, and the result would be P_H (R, h, $<\Theta_m>$, σ) where σ is the standard deviation of the $<\Theta_m>$ observations. However, we have assumed the ACM always executes SHM when it fires, and in doing so we have essentially assumed $\sigma << <\Theta_m>$. Thus, we have a leptokurtic (strongly peaked) distribution function for Θ_m centered on $<\Theta_m>$, and consistent with the SHM assumption we can make an approximation and simply replace Θ_m by $<\Theta_m>$ in the previous equations.

IV. RESULTS AND CONCLUSIONS

We now discuss the sensitivity of P_H to the average maximum amplitude $<\Theta_m>$, the target dimension h, and the mode of oscillation (simple versus conical). Figures 7, 8, and 9 were developed using available polynomial expansions for the complete elliptic integrals[1] and straightforward Riemann summation to perform the integrals in equations (28). As examples we have selected $<\Theta_m> = 1^o$, 4^o, 7^o, and 10^o and have arbitrarily restricted our curves to ranges $R \leq 30m$.[2]

Figure 7 shows four $P_H^S(R)$ versus R curves, one for each $<\Theta_m>$, plotted on a log-linear scale. The mode of oscillation is that of a simple pendulum and h=1m. The curves begin to drop from unity at $R=h/\sin(\pi/3)$ and as R increases, the spread between the four curves also increases and quickly becomes substantial. For example, at R=30m, the $<\Theta_m> = 1^o$ and $<\Theta_m> = 4^o$ curves differ by a factor of 1.8, and the 1^o and 10^o curves differ by a factor of 3.6. The slight "knees" circled in the 4^o, 7^o, and 10^o curves mark the points at which $<\Theta_m> = h/R$; $P_H^S(R)$ falls off slightly more rapidly once $<\Theta_m>$ exceeds h/R.

In figure 8, the sensitivity of P_H to the mode of oscillation is illustrated. Both $P_H^S(R)$ and $P_H^C(R)$ versus R are plotted for $<\Theta_m> = 10^o$ and h=1m. Again there is a rapidly growing spread as R increases. At R=30m the curves differ by a factor of 2.8. Not shown are the $<\Theta_m> = 1^o$, 4^o, and 7^o examples; for these cases the curves differ by factors of 1.04, 2.1, and 2.4, respectively.

---

1. Abamowitz and Stegun, equation (17.3.34), p. 591, and equation (17.3.36), p. 592.

2. At small R the submunition can conceivably collide with the target before firing; this fact is not accounted for in the numerical examples.

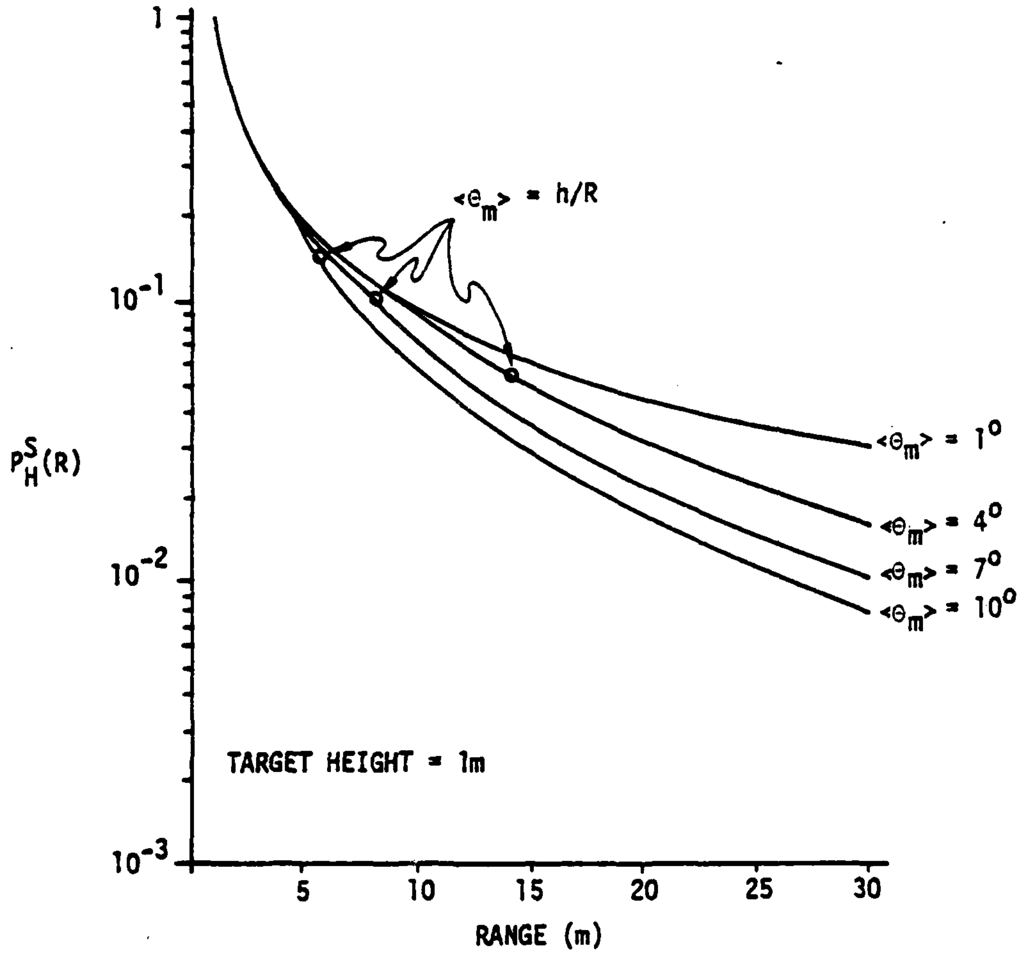

Figure 7. $P_H^S(R)$ Versus R - The Simple Pendulum Mode

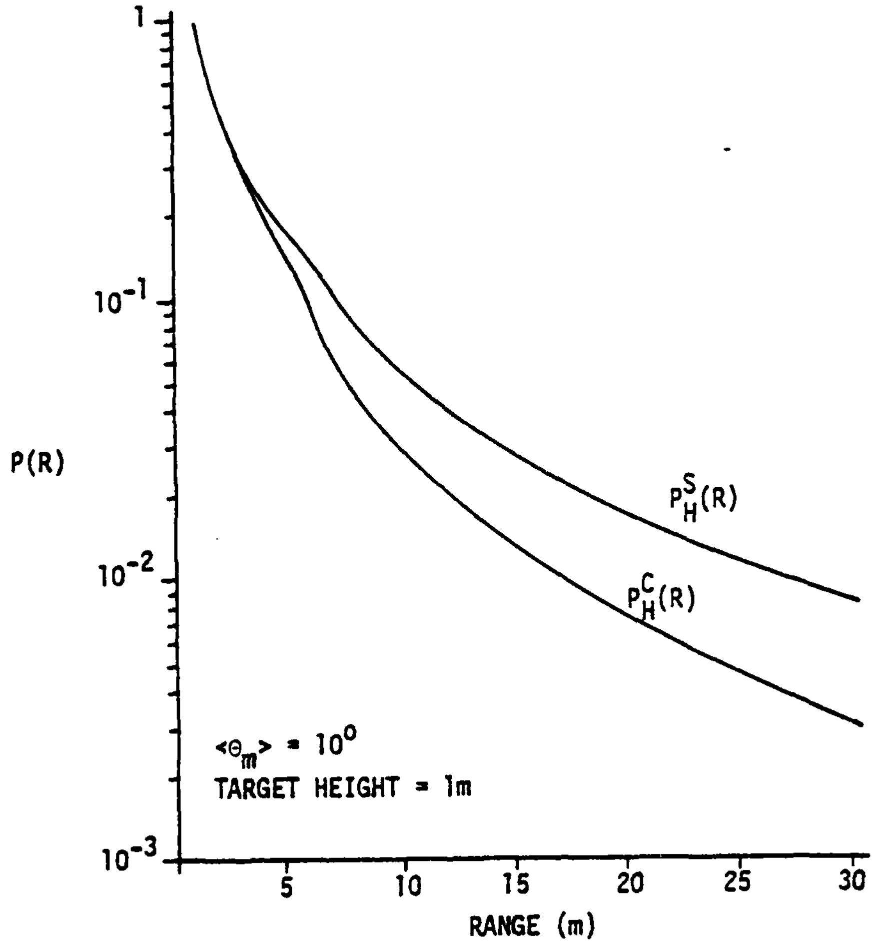

Figure 8. Simple Pendulum Mode Versus Conical Pendulum Mode

Finally, figure 9 shows the sensitivity of P_H^S to h. Examples are plotted which correspond to $<\Theta_m> = 1^o$ and 10^o, and h=1m and 1.5m. The spreading between the $<\Theta_m> = 1^o$ and $<\Theta_m> = 10^o$ curves also increases rapidly for h=1.5m, but at R=30m, the curves differ by a factor of 2.7. This is less than the factor of 3.6 for h=1m but is nevertheless quite subtantial.

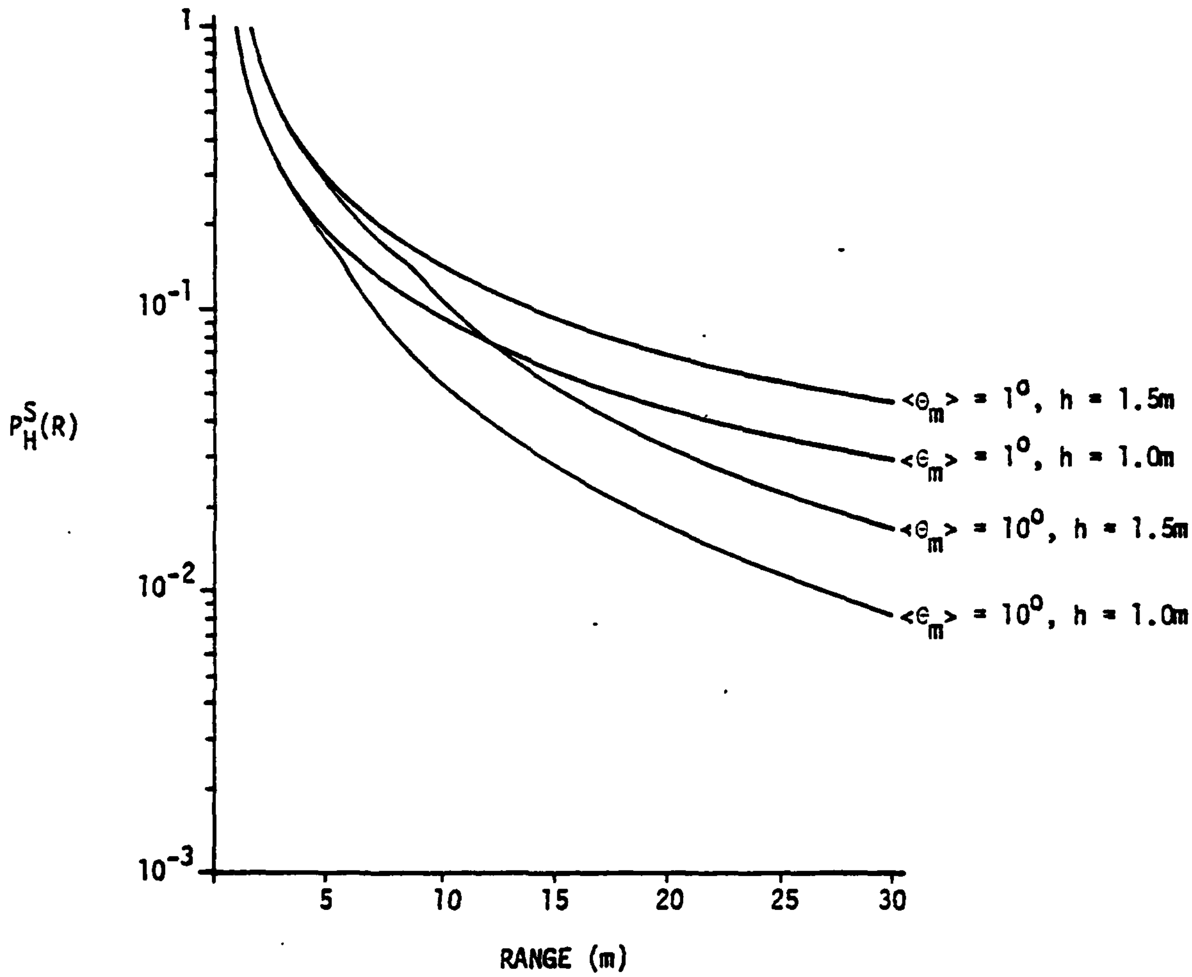

Figure 9. $P_H^S(R)$ Versus R - Sensitivity to Target Dimensions

The discussion thus far leads pointedly to a single conclusion: Against targets that the ACM would typically encounter on the battlefield, the probability of hit is sensitive to the average maximum amplitude. The sensitivity becomes more pronounced with larger $<\Theta_m>$, especially with regard to a target at a large range. The sensitivity is further accentuated if the mode of oscillation deviates from that of a simple pendulum.

At this time we point out that we have treated only P_H and not the probability of kill P_K, for P_K is a product of two factors -- (1) the probability of kill given a hit $P(K|H)$, which depends on target characteristics, and (2) the probability of hit P_H:

$$P_K = P(K|H)\, P_H \tag{29}$$

An analysis of ACM performance in a realistic scenario must therefore also include $P(K|H)$. Nevertheless, we can still conclude that a highly effective and reliable stablizing mechanism, or possibly a homing device, is required in the ACM design to avoid the risk of severe degradation of P_H and therefore P_K.

APPENDIX A

DISTRIBUTION FUNCTION OF THE PERTURBED CONICAL PENDULUM

Although the equation of motion (6) of a spherical pendulum executing SHM cannot be solved in closed form, we nevertheless can investigate a certain restricted case analytically; namely, the perturbed conical pendulum. For this case there are both a maximum amplitude Θ_m and a minimum amplitude Θ_o such that

$$\Theta_m - \Theta_o \approx 0 \tag{A1}$$

By use of condition (A1) and a Taylor series expansion, we will convert the equation of motion (6) into a differential equation for Θ that can easily be solved. We will then convert the solution into the distribution function $\tilde{f}_C(\Theta)d\Theta$ of the perturbed conical pendulum.

The equation of motion of the spherical pendulum executing SHM is

$$\frac{d^2\Theta}{dt^2} + \omega^2\Theta - \frac{C^2}{\Theta^3} = 0 \tag{A2}$$

Consistent with condition (A1), we transfer the time dependence of Θ over to a perturbation variable $\eta \ll \Theta_m$:

$$\Theta(t) = \Theta_m - \eta(t) \tag{A3}$$

Then a Taylor expansion to first order in η gives

$$\frac{1}{\Theta^3} = \frac{1}{(\Theta_m - \eta)^3} \approx \frac{1}{\Theta_m^3}\left(1 + \frac{3\eta}{\Theta_m}\right) \tag{A4}$$

Substitution of equations (A3) and (A4) into equation (A2) gives a second-order linear differential equation for $\eta(t)$:

$$\frac{d^2\eta}{dt^2} + \Omega_1^2 \eta - \Omega_2^2 \Theta_m = 0 \tag{A5}$$

where

$$\Omega_1^2 = \omega^2 + 3\left(\frac{C}{\Theta_m^2}\right)^2 \tag{A5a}$$

$$\Omega_2^2 = \dot{\omega}^2 - \left(\frac{C}{\Theta_m^2}\right)^2 \tag{A5b}$$

To solve equation (A5), we demand that the following boundary conditions hold:

$$\eta(t=o) = 0 \tag{A6a}$$

$$\frac{d\,\eta\,(t=o)}{dt} = 0 \tag{A6b}$$

The physically meaningful solution satisfying these conditions is

$$\eta(t) = \frac{\Omega_2^2\,\Theta_m}{\Omega_1^2}\left[1 - |\cos\,(\Omega_1 t)|\right] \tag{A7}$$

Thus, the perturbation varies periodically with a period $\tau = 2\pi/\Omega_1$.

There is still one unknown constant, C, to evaluate, and therefore we need one more boundary condition. Consistent with the boundary conditions (6), we demand that

$$\eta(t = \tau/4) = \Theta_m - \Theta_o \tag{A8}$$

This gives from equation (A7)

$$\frac{\Omega_2^2\,\Theta_m}{\Omega_1^2} = \Theta_m - \Theta_o$$

and by substitution of equations (A5), we have

$$\left[\omega^2 - \left(\frac{C}{\Theta_m^2}\right)^2\right]\Theta_m = \left[\omega^2 + 3\left(\frac{C}{\Theta_m^2}\right)^2\right](\Theta_m - \Theta_o) \qquad \text{(A9)}$$

from which

$$C = \Theta_m^2\,\omega\sqrt{\frac{\Theta_o}{\Theta_m + 3(\Theta_m-\Theta_o)}} \qquad \text{(A10)}$$

The expressions for Ω_1^2 and Ω_2^2 now reduce to

$$\Omega_1^2 = \frac{4\Theta_m\omega^2}{\Theta_m + 3(\Theta_m-\Theta_o)} \qquad \text{(A11a)}$$

$$\Omega_2^2 = \frac{4(\Theta_m-\Theta_o)\omega^2}{\Theta_m + 3(\Theta_m-\Theta_o)} \qquad \text{(A11b)}$$

and the expression for $\Theta(t)$ reduces to

$$\Theta(t) = \Theta_o + (\Theta_m - \Theta_o)\,|\cos(\Omega_1 t)| \qquad \text{(A12)}$$

If we rewrite equation (A12) in the form

$$\Theta(t) = \Theta_o + (\Theta_m - \Theta_o)\,|\sin(\Omega_1 t + \frac{\pi}{2})| \qquad \text{(A13)}$$

then it is straightforward to show

$$dt = \frac{\tau}{2\pi} \frac{d\theta}{\sqrt{(\theta_m - \theta_o)^2 - (\theta - \theta_o)^2}} \tag{A14}$$

The distribution function $\tilde{f}_C(\theta)d\theta$ then follows in the same way as equation (12) of section II:

$$\tilde{f}_C(\theta)d\theta = \frac{1}{2\pi} \frac{d\theta}{\sqrt{(\theta_m^2 - \theta_o^2) - (\theta - \theta_o)^2}} \tag{A15}$$

This distribution function can be folded into the calculations in section *III* directly, except now two observed quantities are involved -- $<\theta_m>$ and $<\theta_o>$.

APPENDIX B

THE CALCULATION OF P_H (R, h, Θ)

This appendix details the calculation that carries equation (25) over into equations (26). Equation (25) is

$$P_H\ (R,\ h,\ \Theta\ ) = \frac{6}{\pi^2}\int_0^{\arcsin\ (h/R\Theta)} \arccos\left[\sqrt{\frac{1-(h/R)^2}{1-\sin^2\Theta\ \sin^2\phi}}\right] d\phi \qquad (B1)$$

To simplify the notation, we rewrite this in terms of α and ϕ_{CR}:

$$P_H\ (\alpha,\ \Theta,\ \phi_{CR}) = \frac{6}{\pi^2}\int_0^{\phi_{CR}} \arccos\left[\frac{\cos\ (\alpha/2)}{\sqrt{1-\sin^2\Theta\ \sin^2\phi}}\right] d\phi \qquad (B2)$$

An integration by parts gives

$$P_H\ (\alpha,\ \Theta,\ \phi_{CR}) = \frac{6}{\pi^2}\left\{\phi_{CR}\ \arccos\left[\frac{\cos\ (\alpha/2)}{\sqrt{1-\sin^2\Theta\sin^2\phi_{CR}}}\right] + \right. \qquad (B3)$$

$$\left. + \cos\ (\alpha/2)\int_0^{\phi_{CR}} \frac{\phi\sin\phi\cos\phi\ \sin^2\Theta d\phi}{(1-\sin^2\Theta\ \sin^2\phi)\ \sqrt{\sin^2(\alpha/2)\ -\ \sin^2\Theta\sin^2\phi}}\right\}$$

We now use the SHM assumption to reduce the remaining integral which we call "I":

$$I\ (\alpha,\ \Theta,\ \phi_{CR}) = \int_0^{\phi_{CR}} \frac{\Theta^2\phi\sin\phi\cos\phi d\phi}{\sqrt{\sin^2(\alpha/2)\ -\ \Theta^2\ \sin^2\phi}} \qquad (B4)$$

A change of variables gives

$$I(\alpha, \Theta, \phi_{CR}) = \int_0^{\Theta\sin\phi_{CR}} \frac{x \arcsin(x/\Theta)\, dx}{\sqrt{\sin^2(\alpha/2) - x^2}} \tag{B5}$$

and an integration by parts gives

$$\begin{aligned} I(\alpha, \Theta, \phi_{CR}) = & - \phi_{CR} \sqrt{\sin^2(\alpha/2) - \Theta^2 \sin^2\phi_{CR}} + \\ & + \int_0^{\Theta\sin\phi_{CR}} \sqrt{\frac{\sin^2(\alpha/2) - x^2}{\Theta^2 - x^2}}\, dx \end{aligned} \tag{B6}$$

To reduce the remaining integral we again change variables and introduce a new parameter

$$k \equiv \Theta \csc(\alpha/2) = R\Theta/h \tag{B7}$$

which results in

$$\begin{aligned} I(\alpha, \Theta, \phi_{CR}) = & -\phi_{CR} \sin(\alpha/2) \sqrt{1-k^2 \sin^2\phi_{CR}} + \\ & + \sin(\alpha/2) \int_0^{\sin\phi_{CR}} \sqrt{\frac{1-k^2y^2}{1-y^2}}\, dy \end{aligned} \tag{B8}$$

The remaining integral is the elliptic integral of the second kind[1], and so we have the result

$$P_H(\alpha, \Theta, \phi_{CR}) = \frac{6}{\pi^2}\left\{\phi_{CR} \arccos\left[\frac{\cos(\alpha/2)}{\sqrt{1-\sin^2\Theta\sin^2\phi_{CR}}}\right] + \right.$$

$$\left. + \frac{\sin\alpha}{2}\left[E(\phi_{CR}, k) - \phi_{CR}\sqrt{1-k^2 \sin^2 \phi_{CR}}\right]\right\} \cdot \tag{B9}$$

For $\phi_{CR} = \pi/2$, this reduces to

$$P_H(\alpha, \Theta, \phi_{CR} = \pi/2) = \frac{3}{\pi}\left\{\arccos\left[\frac{\cos(\alpha/2)}{\cos\Theta}\right] + \right.$$

$$\left. + \frac{\sin\alpha}{2}\left[\frac{2}{\pi} E(k) - \sqrt{1-k^2}\right]\right\} \tag{B10}$$

where $E(k)$ is the complete elliptic integral of the second kind.[2] Equation (26a) follows immediately upon substituting the expression for k (equation (B7)) and for α (equation (15)) into equation (B10). Note that for $\Theta=0$, equation (B10) reduces to equation (16).

To recover equation (26b), let us first look at $E(\phi_{CR}, k)$:

$$E(\phi_{CR}, k) \equiv \int_0^{\sin\phi_{CR}} \sqrt{\frac{1-k^2y^2}{1-y^2}}\, dy \tag{B11}$$

---

1. Gradshteyn, I.S., and Ryzhik, I.W., Table of Integrals, Series, and Products, p. 905, Academic Press, Inc., New York, 1965. (Short Title - Gradshteyn and Ryzhik) Note that Gradshteyn and Ryzhik's k^2 is Abramowitz and Stegun's m.

2. Gradshteyn and Ryzhik, p. 905.

According to equation (19),

$$\sin\phi_{CR} = h/R\Theta = 1/k \tag{B12}$$

Thus, equation (B11) becomes

$$E(1/k,\ k) = \int_0^{1/k} \sqrt{\frac{1-k^2y^2}{1-y^2}}\ dy \tag{B13}$$

A change of variables gives

$$E(1/k,\ k) = \int_0^{1} \sqrt{\frac{1 - z^2}{k^2 - z^2}}\ dz \tag{B14}$$

This integral can be solved in closed form; the solution is[1]:

$$E(1/k,\ k) = k\ E(1/k) - \left(\frac{k^2-1}{k}\right) K(1/k) \tag{B15}$$

where K is the complete elliptic integral of the first kind.[2] If we now substitute equations (18) and (B15) into equation (B9), we find (remembering $\sin\Theta \approx \Theta$)

$$P_H\ (\alpha,\ \Theta, \phi_{CR} < \pi/2) = \frac{3\ \sin\alpha}{\pi^2}\left[k\ E(1/k) - \left(\frac{k^2-1}{k}\right) K(1/k)\right] \tag{B16}$$

Equation (26b) now follows immediately upon substituting the expressions for k and α.

---

1. Gradshteyn and Ryzhik, equation (3.169.9), p. 276.
2. Gradshteyn and Ryzhik, p. 905.

APPENDIX C

THE CALCULATION OF $P_H\ (R,\ h,\ \Theta_m)$

This appendix details the calculations that carry equation (27) over into equations (28). We begin by deriving equation (28a). Substituting (12) and (26a) into equation (27) gives

$$P_H^S\ (R,\ h,\ \Theta_m \le h/R) = \frac{6}{\pi^2} \int_0^{\Theta_m} \arccos\left[\frac{\sqrt{1-(h/R)^2}}{\cos\Theta}\right] \frac{d\Theta}{\sqrt{\Theta_m^2 - \Theta^2}}$$

$$+ \frac{12h}{\pi^3 R} \sqrt{1-\left(\frac{h}{R}\right)^2} \left[\int_0^{\Theta_m} E\left(\frac{R\Theta}{h}\right) \frac{d\Theta}{\sqrt{\Theta_m^2 - \Theta^2}} - \right. \tag{C1}$$

$$\left. - \frac{\pi}{2} \int_0^{\Theta_m} \sqrt{1-\left(\frac{R\Theta}{h}\right)^2} \frac{d\Theta}{\sqrt{\Theta_m^2 - \Theta^2}}\right]$$

An integration by parts of the first integral gives

$$I_1^S = \int_0^{\Theta_m} \arccos\left[\frac{\cos\ (\alpha/2)}{\cos\ \Theta}\right] \frac{d\Theta}{\sqrt{\Theta_m^2 - \Theta^2}}$$

$$= \frac{\pi}{2} \arccos\left[\frac{\cos\ (\alpha/2)}{\cos\ \Theta_m}\right] + \int_0^{\Theta_m} \frac{\sin\ \Theta \arcsin\ (\Theta/\Theta_m)\ d\Theta}{\cos\Theta \sqrt{\cos^2\Theta\ \sec^2(\alpha/2) - 1}} \tag{C2}$$

Using the SHM approximation, this becomes

$$I_1^S = \frac{\pi}{2} \arccos\left[\frac{\cos\ (\alpha/2)}{\cos\Theta_m}\right] + \cot\ (\alpha/2) \int_0^{\Theta_m} \frac{\Theta \arcsin\ (\Theta/\Theta_m)\ d\Theta}{\sqrt{1 - \Theta^2\ \csc^2\ (\alpha/2)}} \tag{C3}$$

Another integration by parts gives

$$I_1^S = \frac{\pi}{2} \arccos\left[\frac{\cos(\alpha/2)}{\cos\Theta_m}\right] - \frac{\pi}{4}\sin\alpha\sqrt{1-\Theta_m^2\csc^2(\alpha/2)}$$

$$+ \frac{\sin\alpha}{2}\int_0^{\Theta_m}\sqrt{\frac{1-\Theta^2\csc^2(\alpha/2)}{\Theta_m^2-\Theta^2}}\, d\Theta \tag{C4}$$

Equation (28a) follows immediately upon substituting this into equation (C1).

We now turn to equation (28b). For the case $\Theta_m > \alpha/2$ two terms arise. The first corresponds to replacing Θ_m by $\alpha/2$ in equation (C1), and the second corresponds to an integration of equation (26b) over $f_S(\Theta)d\Theta$ from $\alpha/2$ to Θ_m. Thus, we have

$$P_H^S(R, h, \Theta_m > \alpha/2) = \frac{12h}{\pi^3 R}\sqrt{1-\left(\frac{h}{R}\right)^2}\left\{\int_0^{\frac{h}{R}} E\left(\frac{R\Theta}{h}\right)\frac{d\Theta}{\sqrt{\Theta_m^2-\Theta^2}}\right.$$

$$\left. + \int_{\frac{h}{R}}^{\Theta_m}\left[\frac{R\Theta}{h}E\left(\frac{h}{R\Theta}\right) - \left(\frac{R\Theta}{h} - \frac{h}{R\Theta}\right)K\left(\frac{h}{R\Theta}\right)\right]\frac{d\Theta}{\sqrt{\Theta_m^2-\Theta^2}}\right\} \tag{C5}$$

A change of variables to $x = \arcsin(\Theta/\Theta_m)$ results in equation (28b). The second integral can be reduced to two integrals involving only E and not K in their kernels, but these still require numerical integration.

Note that for the case $\Theta_m = h/R$ equations (28a) and (28b) match:

$$P_H^S(R, h, \Theta_m = h/R) = \frac{12h}{\pi^3 R}\sqrt{1-\left(\frac{h}{R}\right)^2}\int_0^1\frac{E(x)}{\sqrt{1-x^2}}\,dx \tag{C6}$$

The integral has a closed form solution,[1] and the result is

$$P_H^S\,(R,\ h,\ \Theta_m = h/R) = \frac{3}{2\pi^3}\left[4\,K^2\left(\frac{\sqrt{2}}{2}\right) + \frac{\pi^2}{K^2\left(\frac{\sqrt{2}}{2}\right)}\right] \times$$

$$\times\,\frac{h}{R}\,\sqrt{1 - \left(\frac{h}{R}\right)^2} \tag{C7}$$

$$\approx 0.8\ \Theta_m\sqrt{1 - \Theta_m^2}$$

1. Gradshteyn and Ryzhik, equation (6.151), p. 638.

LIST OF ACRONYMS AND SYMBOLS

ACM	Antiarmor Cruise Munition
C	Integral of the Spherical Pendulum Motion (see equation (4b))
E	Complete Elliptic Integral of the Second Kind
f	Distribution Function
$f_S(\Theta)$	Distribution Function of the Orientation Θ of the Simple Pendulum
$f_C(\Theta)$	Distribution Function of the Orientation Θ of the Conical Pendulum
$\tilde{f}_C(\Theta)$	Distribution Function of the Orientation Θ of the Perturbed Conical Pendulum
g	Gravitational Acceleration
h	Radius of Spherical Target
I	Auxiliary Integral (see equation (B4))
I_1, I_2, I_3	Integrals Comprising P_H^S (see equations (28))
I_1^S	Auxiliary Integral (see equation (C2))
k	Auxiliary Parameter (see equation (B7))
K	Complete Elliptic Integral of the First Kind
ℓ	Length of Parachute
L	Lagrangian
m	Mass of Submunition
P_H	Probability of Hit
P_H^C	Probability of Hit of the Conical Pendulum Model
P_H^S	Probability of Hit of the Simple Pendulum Model
P_K	Probability of Kill
$P(K\|H)$	Probability of Kill Given a Hit
R	Range
SHM	Simple Harmonic Motion
t	Time
T	Kinetic Energy
v	Velocity
V	Potential Energy
WAAM	Wide-Area Antiarmor Munition
α	Maximum Angle Subtended by Target (see equation (15))
β	Angle Subtended by Exposed Target (see equation (21))

γ	Azimuth of Spherical Pendulum
η	Perturbation Amplitude (see equation (A3))
Θ	ACM Orientation (see figure 2)
Θ_m	Maximum Amplitude of Orientation
$\langle\Theta_m\rangle$	Average Maximum Amplitude of Orientation
Θ_o	Minimum Amplitude of Orientation
σ	Standard Deviation
τ	Period of Pendulum
ϕ	Azimuth of Target
χ	Auxiliary Angle (see equations (20))
ψ	Auxiliary Angle (see equations (20))
ω	Circular Frequency of Pendulum
Ω_1	Circular Frequency of Perturbed Conical Pendulum
Ω_2	Auxiliary Frequency (see equation (A5b))